

Extra-Dimensions effects on the fermion-induced quantum energy in the presence of a constant magnetic field

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Abstract

We consider a $U(1)$ gauge field theory with fermion fields (or with scalar fields) that live in a space with δ extra compact dimensions, and we compute the fermion-induced quantum energy in the presence of a constant magnetic field, which is directed towards the x_3 axis. Our motivation is to study the effect of extra dimensions on the asymptotic behavior of the quantum energy in the strong field limit ($eB \gg M^2$), where $M = 1/R$. We see that the weak logarithmic growth of the quantum energy for four dimensions, is modified by a rapid power growth in the case of the extra dimensions.

1 Introduction

The computation of the fermion-induced quantum energy in the presence of a constant magnetic field (or Heisenberg-Euler lagrangian), is a topic that has attracted the attention of authors from the early time of quantum electrodynamics [1, 2, 3]. In addition, the case of three dimensions has been studied in Ref. [4]. It is worth mentioning that inhomogeneous magnetic fields have also been studied, analytically and numerically [5, 6, 7, 8]. Finally we note that approximation tools such as the derivative expansion are also available [9].

As it is believed, particle field theories like Standard Model are embedded in more fundamental field theories, which may be string theories. It is well known that string theories are formulated in higher dimensional manifolds. For this reason, in recent years, there has been a great interest for particle models with extra compact dimensions (see for example [10, 11, 12, 13, 14, 15]).

In the framework of the above discussion it would be interesting to reconsider classical topics, like Heisenberg-Euler Lagrangian (for QED), in the case of models with extra dimensions. The simplest way to extend QED (or a $U(1)$ gauge field theory with fermions) in this

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direction is to add one or more extra compact dimensions with radius $R = 1/M$, assuming that both the gauge and the fermionic fields live in the bulk. We note that M is the mass scale of the Kaluza-Klein modes.

In this work we will generalize the well known Schwinger formula [3] for the effective energy in the case of δ extra compact dimensions. Our motivation is to study the effect of the extra dimensions on the strong field ($eB \gg M^2$) asymptotic behavior of the quantum part ¹ of the effective energy. We see that the weak logarithmic growth of the quantum energy, in the case of four dimensions, is modified by a rapid power growth (see Eqs. (37),(43) and (45) below) in the case of the extra dimensions. The reason is that in the strong field limit ($eB \gg M^2$) the extra dimensions lose their compact structure and behave as if they were noncompact.

Previous works aimed at the study of extra noncompact dimensions with external magnetic fields can be found in Ref. [16]. However, the topics which are covered in these works are different to what this paper aims at.

The question that arises is whether the above mentioned model can be assumed as an extra-dimension extension of QED. Note that the smallest scale of extra dimensions that has been assumed is $M \sim 1\text{TeV}$ (for a specification of bounds on M see Ref. [17]), according to the scenario of Refs. [10, 11, 12]. It is obvious that for $\sqrt{eB} \sim 1\text{TeV}$ (or $B \sim 10^{26}\text{G}$) the QED is not valid. Thus, we will use this model for an understanding of the effects of extra dimensions to the effective energy, but we can not use it in order to extract trustworthy physical results (see also the discussion in conclusions).

2 Quantum energy for five dimensions

In this section we study the case of five dimensions, and later we will generalize our results for more extra dimensions.

The partition function of a U(1) gauge field theory with fermions, in dimension D=5, reads

$$Z = \int_{b.c} \mathcal{D}A \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^D x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i \not{D} - m_f) \Psi + \mathcal{L}_{hd})} \quad (1)$$

where $\int d^D x = \int_0^{2\pi R} dx_4 \int d^4 x$ and x_4 is the extra dimension, which is assumed to be compactified on a circle of radius $R = 1/M$ (M is the scale of the Kaluza-Klein modes).

In addition, we have assumed periodic boundary conditions for the fermion and gauge fields, namely $\Psi(x, x_4) = \Psi(x, x_4 + 2\pi R)$ and $A_\mu(x, x_4) = A_\mu(x, x_4 + 2\pi R)$, where $x = (x_0, x_1, x_2, x_3)$.

At this point we remind the reader that the model we examine is nonrenormalizable. However, in this paper, we will assume the above model as a low energy effective field theory which is valid up to large physical cut-off Λ_{Ph} , above which a new well defined theory emerges. In particular, we assume that this effective field theory has been obtained from an original fundamental field theory by integrating out all higher momenta and heavy particles above the physical cut-off Λ_{Ph} . Thus, the model we examine is viewed as a Wilsonian effective

¹The effective energy E_{eff} is equal to the classical energy E_{class} plus a quantum energy part E_Q , which is induced by the fermions, or $E_{eff} = E_{class} + E_Q$.

field theory [18, 19, 20]. For more details and criticism on effective field theories with extra compact dimensions, see Ref. [20].

The lagrangian \mathcal{L}_{hd} incorporates all the possible higher dimension operators with dimension $2 + D$ or higher, which respect the symmetries of the above model. For example the lagrangian \mathcal{L}_{hd} should be Gauge invariant. However, the precise form of \mathcal{L}_{hd} can not be determined, as the original renormalizable fundamental theory that our model has come from, is not known. In general, we will ignore the contributions of the higher dimension operators, or we will drop the lagrangian \mathcal{L}_{hd} from the path integral of Eq. (1). However, in section 4.2, it is necessary to accept the existence of a $2D$ dimension operator of the form $w_D/2 \int d^D x (F_{\mu\nu} F^{\mu\nu})^2$, in order to incorporate the cut-off dependent part of the quantum energy for $D = 8, 9, 10$ (for details see section 4.2).

The effective action $S_{eff}[A]$ is defined by the equation

$$e^{iS_{eff}[A]} = \int_{b.c} \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^D x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\mathcal{D} - m_f)\Psi)} = e^{i(-\frac{1}{4} \int d^D x F_{\mu\nu} F^{\mu\nu} + S_Q[A])} \quad (2)$$

The quantum part of the effective action $S_Q[A]$ is obtained by integrating out the fermionic degrees of freedom in the path integral, so we obtain

$$S_Q[A] = \frac{1}{i} \ln \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{i \int d^D x \bar{\Psi}(i\mathcal{D} - m_f)\Psi} = \frac{1}{i} Tr \ln(i\mathcal{D} - m_f) \quad (3)$$

where $\mathcal{D} = \Gamma^\mu(\partial_\mu - ie_5 A_\mu)$ ($\mu = 0, 1, 2, 3, 4$), and e_5 is the five dimensional coupling constant. The gamma matrices for the five dimensional case are four dimensional matrices which satisfy the Clifford algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \mathcal{I}_{4 \times 4}$. A representation for Γ_μ is obtained from the usual representation γ_μ of the four dimensional QED, by setting $\Gamma_\mu = \gamma_\mu$ for $\mu = 0, 1, 2, 3$ and $\Gamma_4 = i\gamma_5$.

It is well known that for odd dimensions, a parity² violating term (Chern-Simons term³) is induced by the quantum corrections. However this term is zero in the case of the constant magnetic field.

Thus we will concentrate on the parity invariant term of the quantum action which reads:

$$S_Q[A] = \frac{1}{2i} Tr \ln(\mathcal{D}^2 + m_f^2) \quad (4)$$

Instead of the effective action, it is more convenient for our purposes to use the quantum part of the effective energy per unit of the three dimensional volume $V = L^3$ (where L is the size of the three dimensional space box) which is given by the following equation

$$E_Q[A] = -\frac{1}{VT} S_Q[A] \quad (5)$$

where T is the total length of time.

²In the case of five dimensions the symmetry of parity can be defined as $x \rightarrow x$ and $x_4 \rightarrow -x_4$. The fermionic field is transformed as $\Psi(x, x_4) \rightarrow \Gamma_4 \Psi(x, -x_4)$, then the kinetic term of the Dirac Lagrangian is invariant under the symmetry of parity, but the mass term violates it.

³The induced Chern-Simons term is of the form $\int d^3 x \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau$, in the case of five dimensions, and it is zero in the case we examine.

In this paper we aim to compute the effective action in the presence of a constant magnetic field which is directed toward the x_3 axis. The vector potential that corresponds to this magnetic field is $A = (0, 0, B_5 x_1, 0, 0)$, where $B_5 = F_{12}$ is the five dimensional field strength. The magnetic field that corresponds to four dimensions is $\vec{B} = B\vec{e}_3$ where $B = B_5\sqrt{2\pi}/M$. Also we define the four dimensional coupling constant e as $e = e_5\sqrt{M/2\pi}$. Note that the product $eB(= e_5B_5)$ is independent of the dimensionality of space, and has dimension of square of mass in natural units.

In what follows we will use the four dimensional quantities. For example, the classical energy per unit of the three dimensional volume can be expressed as

$$E_{class} = \frac{1}{2} \frac{2\pi}{M} B_5^2 = \frac{1}{2} B^2 = \frac{1}{2e^2} (eB)^2 \quad (6)$$

From the integral representation $\ln(a/b) = -\int_0^{+\infty} (ds/s)(e^{-as} - e^{-bs})$, if we use the Eqs. (4) and (5) we obtain:

$$E_Q = -\frac{i}{2TV} \int_0^{+\infty} \frac{ds}{s} e^{-sm_f^2} Tr(e^{-s\mathcal{D}^2} - e^{-s\mathcal{Q}^2}) \quad (7)$$

where we have renormalized by subtracting the effective energy for zero magnetic field.

The trace in the above equation can be written as

$$Tr e^{-s\mathcal{D}^2} = Tr e^{-s(D^2 + \frac{1}{2}e\Sigma^{\mu\nu}F_{\mu\nu})} = Tr e^{-sD^2} tr e^{-s\frac{1}{2}e\Sigma^{\mu\nu}F_{\mu\nu}} \quad (8)$$

where $\Sigma^{\mu\nu} = \frac{i}{2}[\Gamma_\mu, \Gamma_\nu]$.

By using the equation

$$e^{-s\frac{1}{2}e\Sigma^{\mu\nu}F_{\mu\nu}} = e^{-si\Gamma_1\Gamma_2eB} = \cosh(eBs)\mathcal{I}_{4\times 4} - i\Gamma_1\Gamma_2 \sinh(eBs) \quad (9)$$

we obtain

$$tr e^{-s\frac{1}{2}e\Sigma^{\mu\nu}F_{\mu\nu}} = 4 \cosh(eBs) \quad (10)$$

where we have used the identity $tr\Gamma_1\Gamma_2 = 0$.

The operator D^2 in the presence of the magnetic field is:

$$D^2 = \partial_0^2 - \partial_1^2 - (\partial_2 - ieBx_1)^2 - \partial_3^2 - \partial_4^2 \quad (11)$$

We will compute the trace $Tr e^{-sD^2}$ by using the complete basis of eigenfunctions

$$\Psi(x, x_4) \sim e^{-i\omega x_0} e^{ip_2 x_2} e^{ip_3 x_3} e^{iMm x_4} u(x_1) \quad (m = 0, \pm 1, \pm 2, \dots) \quad (12)$$

The function $u(x_1)$ satisfies the eigenvalue equation

$$(-\partial_1^2 + (p_2 + eBx_1)^2) u(x_1) = E_n u(x_1) \quad (13)$$

and the corresponding eigenvalues E_n are the well known Landau levels:

$$E_n = 2eB(n + \frac{1}{2}) \quad (n = 0, 1, 2, \dots) \quad (14)$$

We remind the reader that in every Landau level corresponds an infinite degeneracy factor with value $eBL^2/2\pi$.

From Eqs. (11),(12),(13) and (14), if we perform a wick rotation $\omega \rightarrow i\omega$, we obtain

$$Tr e^{-s D^2} = i \frac{TL^3 eB}{16\pi^2} \frac{1}{s \sinh(eBs)} \sum_{m=-\infty}^{+\infty} e^{-s M^2 m^2} \quad (15)$$

where we have used the equation $\sum_n e^{-s E_n} = 1/2 \sinh(eBs)$.

By using Eqs. (10) and (15) we obtain

$$E_Q = \frac{eB}{8\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} \right) F(sM^2) \quad (16)$$

where we have set

$$F(s) = \sum_{m=-\infty}^{+\infty} e^{-m^2 s} \quad (17)$$

and we have rendered the integral of the Eq. (16) convergent by introducing an ultraviolet cut-off $^4\Lambda$.

If we use the expansion

$$\coth(z) = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + O(z^5) \quad (18)$$

the Eq. (16) can be written as:

$$\begin{aligned} E_Q &= \frac{eB}{8\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) F(sM^2) \\ &+ \frac{(eB)^2}{24\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s} e^{-s m_f^2} F(sM^2) \end{aligned} \quad (19)$$

The last term, which is cut-off dependent, corresponds to the diagram with one fermionic loop and two external legs ⁵. This term can be incorporated in the classical energy, as is shown bellow

$$\begin{aligned} E_{eff} &= E_{class} + E_Q \\ &= \frac{1}{2} \left(\frac{1}{e^2} + \frac{1}{12\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s} e^{-s m_f^2} F(sM^2) \right) (eB)^2 \\ &+ \frac{eB}{8\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) F(sM^2) \end{aligned} \quad (20)$$

⁴Note that the cut-off Λ , which corresponds to the proper time-method, is not equal to the physical cut-off scale Λ_{Ph} . However, we assume that they are connected by a linear relation $\Lambda_{Ph} = \sqrt{r}\Lambda$ where r is of the order of unity. A value for the parameter r has been determined in Ref. [12], and it is found to be of the order of unity for all the values of δ . In addition, a different study for the parameter r , from the point view of the Wilson renormalization group, can be found in Ref. [14]. However, in this work it is not necessary to introduce the fundamental scale explicitly, and thus we will not deal further with this topic.

⁵This diagram is known as the vacuum polarization diagram, and the external lengths represent the interaction of the fermion with the classical magnetic field.

The renormalized coupling constant e_R is defined as

$$\frac{1}{e_R^2} = \frac{1}{e^2} + \frac{1}{12\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s} e^{-s m_f^2} F(sM^2) \quad (21)$$

Note that the coupling constant e is the bare coupling constant which corresponds to the scale Λ_{Ph} , and e_R is the renormalized coupling constant defined according to the first of the renormalization conditions of Eq. (41) in section 4.2.

If we take that eB is a renormalization group invariant quantity (or $eB = e_R B_R$), we can write

$$E_{eff} = \frac{1}{e_R^2} (e_R B_R)^2 + E_Q^{(R)} \quad (22)$$

where the renormalized quantum part of the effective energy $E_Q^{(R)}$ is cut-off independent. If we set $z = eBs$ in Eq. (20) we obtain

$$E_Q = \frac{(eB)^2}{8\pi^2} \int_0^{+\infty} \frac{dz}{z^2} e^{-\frac{z m_f^2}{eB}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} \right) F\left(\frac{z M^2}{eB}\right) \quad (23)$$

For the sake of simplicity, we have dropped the index (R), otherwise in the above equation we should write e_R , B_R and $E_Q^{(R)}$.

We note that $F(zM^2/eB) \rightarrow 1$ as $eB/M^2 \ll 1$. Thus the Eq. (23) is reduced to the usual expression for the quantum energy in the case of four dimensions, as is required.

3 Strong field limit ($eB \gg M^2$)

We remind the reader that the strong field asymptotic formula for the case of four dimensions reads:

$$E_Q^{asympt} = -\frac{(eB)^2}{24\pi^2} \ln\left(\frac{eB}{m_f^2}\right) \quad (24)$$

See for example Refs. [22, 21].

Now if we use the Poisson formula we can prove that

$$F(s) = \sum_{m=-\infty}^{+\infty} e^{-m^2 s} = \sqrt{\frac{\pi}{s}} \sum_{r=-\infty}^{+\infty} e^{\frac{-\pi^2 r^2}{s}} \quad (25)$$

and from this equation we obtain $F(s) \sim \sqrt{\frac{\pi}{s}}$ as $s \rightarrow 0$.

In order to isolate the asymptotic behavior of Eq. (19) for $s \rightarrow 0$, we define the function

$$L_1(s) = F(s) - \sqrt{\frac{\pi}{s}} \quad (26)$$

which has the identity: $\lim_{s \rightarrow 0} s^{-n} L_1(s) = 0$ for every integer n . This asymptotic behavior for $s \rightarrow 0$ has been confirmed numerically by plotting the function $s^{-n} L_1(s)$ for several values

of n . Note that due to this identity the first two integrals, in Eq. (27) below, are rendered convergent.

If we use the Eqs. (26) and (23) we obtain

$$E_Q = \frac{eB}{8\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} \right) L_1(sM^2) - \frac{(eB)^2}{24\pi^2} \int_0^{+\infty} \frac{ds}{s} e^{-s m_f^2} L_1(sM^2) \\ + \frac{eB}{8\pi^{3/2}M} \int_0^{+\infty} \frac{ds}{s^{5/2}} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) \quad (27)$$

In order to study the asymptotic behavior of E_Q in the strong magnetic field limit $eB \gg M^2$ we will study separately the three integrals that appear in Eq. (27). It will be convenient to call them $E_Q^{(1)}$, $E_Q^{(2)}$ and $E_Q^{(3)}$ respectively.

If we set $y = sM^2$ in the first integral of Eq. (27) we obtain

$$E_Q^{(1)} = \frac{(eB)^2}{8\pi^2} \left(\frac{eB}{M^2} \right)^{-1} \int_0^{+\infty} \frac{dy}{y^2} e^{-y \frac{m_f^2}{M^2}} \left(\coth\left(\frac{eBy}{M^2}\right) - \frac{1}{eBy/M^2} \right) L_1(y) \quad (28)$$

and for $\frac{eB}{M^2} \rightarrow +\infty$ we find

$$E_Q^{(1)} \rightarrow \frac{(eB)^2}{8\pi^2} \left(\frac{eB}{M^2} \right)^{-1} \int_0^{+\infty} \frac{dy}{y^2} e^{-y \frac{m_f^2}{M^2}} L_1(y) \quad (29)$$

Similarly if we set $z = eBs$ in the third integral of Eq. (27) we obtain

$$E_Q^{(3)} = \frac{(eB)^2}{8\pi^{3/2}} \sqrt{\frac{eB}{M^2}} \int_0^{+\infty} \frac{dz}{z^{5/2}} e^{-z \frac{m_f^2}{eB}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} \right) \quad (30)$$

and for $eB \gg M^2 \gg m_f^2$ we see that

$$E_Q^{(3)} \rightarrow \frac{(eB)^2}{8\pi^{3/2}} \sqrt{\frac{eB}{M^2}} \int_0^{+\infty} \frac{dz}{z^{5/2}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} \right) = -\frac{c_1(eB)^2}{8\pi^{3/2}} \sqrt{\frac{eB}{M^2}} \quad (31)$$

where c_1 is a positive constant. For the numerical value of c_1 see Table 1.

The integral

$$E_Q^{(2)} = -\frac{(eB)^2}{24\pi^2} \int_0^{+\infty} \frac{ds}{s} e^{-s m_f^2} L_1(sM^2)$$

is obviously proportional to $(eB)^2$ and no further analysis is needed.

If we compare the equations (28) and (30) we see that $E_Q^{(3)}$ dominates in the strong field limit. Thus, the leading term of the asymptotic formula for the quantum energy E_Q is

$$E_Q^{(asympt)} = -\frac{c_1(eB)^2}{8\pi^{3/2}} \sqrt{\frac{eB}{M^2}} \quad (32)$$

and the next to leading term is given by the integral $E_Q^{(2)}$.

Comparing Eq. (32) with Eq. (24) we see that the logarithmic dependence for four dimensions has been modified with a square root law in the case of five dimensions.

4 More than five dimensions (D=4+ δ)

In this section we aim to generalize the asymptotic formula of Eq. (32) for more than five dimensions. We set $D = \delta + 4$ where δ is the number of the extra compact dimensions. However, if we assume that the number of extra dimensions is restricted by the string theory, it can not exceed the number six (or $1 \leq \delta \leq 6$). Also we assume that the radius $R = 1/M$ is the same for all the extra dimensions.

An obvious modification, in order to extend Eq. (16) for a general number of extra dimensions δ , is to do the replacement $F(sM^2) \rightarrow (F(sM^2))^\delta$. A second modification is to multiply Eq. (16) by $2^{[D/2]-2}$, which is due to the trace of gamma matrices ⁶ $\text{tr} e^{-s\frac{1}{2}e\Sigma^{\mu\nu}F_{\mu\nu}} = 2^{[D/2]} \cosh(eBs)$.

Now it is a straightforward matter to generalize the formula of Eq. (16) in the case of δ extra dimensions

$$E_Q = 2^{[D/2]-2} \frac{eB}{8\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} \right) (F(sM^2))^\delta \quad (33)$$

where $e = e_D(M/2\pi)^{\delta/2}$, $B = B_D(2\pi/M)^{\delta/2}$, and e_D and B_D are the corresponding D dimensional quantities.

Note that for $eB/M^2 \ll 1$ the above equation is not reduced to the corresponding four dimensional expression, as is required. In particular it differs by a factor $2^{[D/2]-2}$. This is due to the fact that the reduced theory contains $2^{[D/2]-2}$ four dimensional Dirac spinors with the same mass term, and not one as happens with the four dimensional model. Thus we can make contact with the four dimensional quantum energy by dividing with the number of Dirac spinors $2^{[D/2]-2}$. A more sophisticated way to solve this problem would be an orbifold model with the appropriate boundary conditions, so that only one Dirac spinor would survive in four dimensions. However in this work we will not perform computations for this case.

An interesting point is that the strong field ($eB \gg M^2$) asymptotic behavior for the effective energy, is independent from the details of the compactification. The reason is that, for $eB \gg M^2$, the extra dimensions lose their compact structure and behave as if they were noncompact. Thus, even if we had assumed another compactification scenario, for example an orbifold model, we would obtain exactly the same result for the effective energy in the strong field limit.

The next step is to subtract and to add back the vacuum polarization diagram, which is responsible for the renormalization of the coupling constant,

$$\begin{aligned} E_Q &= 2^{[D/2]} \frac{eB}{32\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) (F(sM^2))^\delta \\ &+ 2^{[D/2]} \frac{(eB)^2}{96\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s} e^{-s m_f^2} (F(sM^2))^\delta \end{aligned} \quad (34)$$

In addition, the renormalized coupling constant is

$$\frac{1}{e_R^2} = \frac{1}{e^2} + \frac{2^{[D/2]}}{48\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s} e^{-s m_f^2} (F(sM^2))^\delta \quad (35)$$

⁶We remind the reader that the gamma matrices, in a D-dimensional space, have dimensions $2^{[D/2]} \times 2^{[D/2]}$.

4.1 Effective Energy for $\delta = 1, 2, 3$

For $\delta = 1, 2, 3$, the renormalized quantum energy reads:

$$E_Q = 2^{[D/2]} \frac{(eB)^2}{32\pi^2} \int_0^{+\infty} \frac{dz}{z^2} e^{-\frac{z m_f^2}{eB}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} \right) \left(F\left(\frac{z M^2}{eB}\right) \right)^\delta \quad (36)$$

Note that the above integral is convergent only for $\delta = 1, 2, 3$, and the quantities e and B that appear in this integral are the renormalized ones, for which we have dropped the index (R).

The strong field limit is obtained by using exactly the same method, which was presented in the previous section. The corresponding asymptotic formula reads:

$$E_Q^{asympt} = -c_\delta \frac{2^{[D/2]} (eB)^2}{32 \pi^{(4-\delta)/2}} \left(\frac{eB}{M^2} \right)^{\delta/2} \quad (\delta = 1, 2, 3) \quad (37)$$

where

$$c_\delta = - \int_0^{+\infty} \frac{dz}{z^{(\delta+4)/2}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} \right) \quad (\delta = 1, 2, 3) \quad (38)$$

The numerical values of the constant $c_\delta > 0$ are given in Table 1.

In order to prove the asymptotic formula of Eq. (37) we have used the function

$$L_\delta(s) = (F(s))^\delta - \left(\frac{\pi}{s} \right)^{\frac{\delta}{2}} \quad (39)$$

Note that the identity $\lim_{s \rightarrow 0} s^{-n} L_\delta(s) = 0$ is valid also in the case of more than one extra dimensions, and that it has been confirmed numerically.

4.2 Effective Energy for $\delta = 4, 5, 6$

For $\delta = 4, 5, 6$ the integral in Eq. (36) is divergent for $s = 0$. Thus, the cut-off Λ appears explicitly in the expression for the quantum energy. It is possible to isolate the cut-off dependent part by subtracting and adding back the Feynman diagram with one fermion loop and four external legs, or

$$\begin{aligned} E_Q &= 2^{[D/2]} \frac{eB}{32\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) \\ &+ 2^{[D/2]} \frac{eB}{32\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} + \frac{(eBs)^3}{45} \right) \left((F(sM^2))^\delta - 1 \right) \\ &- 2^{[D/2]} \frac{(eB)^4}{1440\pi^2} \int_{1/\Lambda^2}^{+\infty} ds s e^{-s m_f^2} \left((F(sM^2))^\delta - 1 \right) \end{aligned} \quad (40)$$

In this case there is not a classical term in which the cut-off dependent term of the above equation can be incorporated. Of course, this is a problem that is due to the nonrenormalizable character of the model we examine. However, we will overcome this problem by assuming the existence of a higher dimension operator of the form $w_D/2 \int d^D x (F_{\mu\nu} F^{\mu\nu})^2$.

The cut-off dependent term in Eq. (40) can be incorporated in this higher dimension operator. The renormalization conditions (see also Ref. [23]) according to which we separate the cut-off dependent part from the finite part are

$$\frac{1}{e^2} \left[\frac{dE_{eff}}{d\mathcal{F}} \right]_{\mathcal{F}=0} = \frac{1}{e_R^2}, \quad \frac{1}{4} \left[\frac{d^2 E_{eff}}{d\mathcal{F}^2} \right]_{\mathcal{F}=0} = w_R - \frac{2^{[D/2]} e^4}{1440\pi^2 m_f^4} \quad (41)$$

where $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, and w_R is the corresponding renormalized four dimensional coupling constant. Thus, the effective energy includes a term of the form $w_R/2 B^4$ where w_R is a free parameter, which can be determined experimentally according to the renormalization conditions of Eq. (41). Note, that we have included the term $-\frac{2^{[D/2]} e^4}{1440\pi^2 m_f^4}$ in Eq. (40), in order to make contact with the four dimensional result for the quantum energy in the weak field limit $eB \ll M^2$.

The renormalized quantum energy consists of the first two terms of Eq. (40):

$$\begin{aligned} E_Q^{(R)} &= 2^{[D/2]} \frac{eB}{32\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} \right) \\ &+ 2^{[D/2]} \frac{eB}{32\pi^2} \int_0^{+\infty} \frac{ds}{s^2} e^{-s m_f^2} \left(\coth(eBs) - \frac{1}{eBs} - \frac{eBs}{3} + \frac{(eBs)^3}{45} \right) \left((F(sM^2))^\delta - 1 \right) \end{aligned} \quad (42)$$

The corresponding strong field asymptotic formula for $\delta = 5, 6$ is

$$E_Q^{asympt} = \frac{c_\delta 2^{[D/2]} (eB)^2}{32 \pi^{(4-\delta)/2}} \left(\frac{eB}{M^2} \right)^{\delta/2} \quad (\delta = 5, 6) \quad (43)$$

where

$$c_\delta = \int_0^{+\infty} \frac{dz}{z^{(4+\delta)/2}} \left(\coth(z) - \frac{1}{z} - \frac{z}{3} + \frac{z^3}{45} \right) \quad (\delta = 5, 6) \quad (44)$$

Note, that from the first term of Eq. (42) we obtain the logarithmic asymptotic behavior of Eq. (24), which corresponds to the case of four dimensions.

For $\delta = 4$ we can show that the strong field asymptotic formula reads:

$$E_Q^{asympt} = \frac{2^{[D/2]} (eB)^2}{1440} \left(\frac{eB}{M^2} \right)^{\delta/2} \ln \left(\frac{eB}{M^2} \right) \quad (\delta = 4) \quad (45)$$

The logarithmic factor $\ln(eB/M^2)$ in the above asymptotic formula, for $\delta = 4$, is due to the logarithmic divergent of the second integral of Eq. (42), as $eB/M^2 \rightarrow +\infty$.

Note, that for $\delta = 4, 5, 6$ the asymptotic formula is positive, contrary to the case of $\delta = 1, 2, 3$.

5 Quantum energy for scalar fields

In this section we will discuss briefly the quantum energy for a U(1) gauge field theory with scalar fields. The path integral in this case is

$$Z = \int_{b.c} \mathcal{D}A \mathcal{D}\phi^* \mathcal{D}\phi e^{i \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^* (D_\mu \phi) - m_s^2 \phi^* \phi \right)} \quad (46)$$

δ	1	2	3	4	5	6
c_δ	0.340	0.122	0.087	-	0.027	0.011
f_δ	0.219	0.091	0.074	-	0.024	0.010

Table 1: The constants c_δ and f_δ for $\delta = 1, 2, 3, 5, 6$.

with boundary conditions $\phi(x, x_4) = \phi(x, x_4 + 2\pi R)$.

The quantum part of the effective action is

$$S_Q[A] = \frac{1}{i} \ln \left(\int_{b,c} \mathcal{D}\phi^* \mathcal{D}\phi e^{-i \int d^D x \phi^* (D^\mu D_\mu + m_s^2) \phi} \right) = -\frac{1}{i} \text{Tr} \ln(D^2 + m_s^2) \quad (47)$$

In a similar way with that of previous sections, we find that the renormalized quantum energy in the case scalar fields is

$$E_Q = -\frac{(eB)^2}{16\pi^2} \int_0^{+\infty} \frac{dz}{z^2} e^{-\frac{z m_s^2}{eB}} \left(\frac{1}{\sinh(z)} - \frac{1}{z} + \frac{z}{6} \right) \left(F\left(\frac{zM^2}{eB}\right) \right)^\delta \quad (48)$$

for $\delta = 1, 2, 3$.

The corresponding strong field asymptotic formula for the quantum energy is

$$E_Q^{asympt} = -f_\delta \frac{(eB)^2}{16 \pi^{(4-\delta)/2}} \left(\frac{eB}{M^2} \right)^{\delta/2} \quad (\delta = 1, 2, 3) \quad (49)$$

where f_δ is a positive constant which is given by the equation

$$f_\delta = \int_0^{+\infty} \frac{dz}{z^{(\delta+4)/2}} \left(\frac{1}{\sinh(z)} - \frac{1}{z} + \frac{z}{6} \right) \quad (\delta = 1, 2, 3) \quad (50)$$

For $\delta = 5, 6$ we obtain

$$E_Q^{asympt} = f_\delta \frac{(eB)^2}{16 \pi^{(4-\delta)/2}} \left(\frac{eB}{M^2} \right)^{\delta/2} \quad (\delta = 5, 6) \quad (51)$$

where f_δ is positive, and it is given by the equation

$$f_\delta = - \int_0^{+\infty} \frac{dz}{z^{(4+\delta)/2}} \left(\frac{1}{\sinh(z)} - \frac{1}{z} + \frac{z}{6} - \frac{7z^3}{360} \right) \quad (\delta = 5, 6) \quad (52)$$

For the numerical values of the constants f_δ see table 1.

Finally, for $\delta = 4$ we obtain

$$E_Q^{asympt} = \frac{7(eB)^2}{5760} \left(\frac{eB}{M^2} \right)^{\delta/2} \ln\left(\frac{eB}{M^2}\right) \quad (\delta = 4) \quad (53)$$

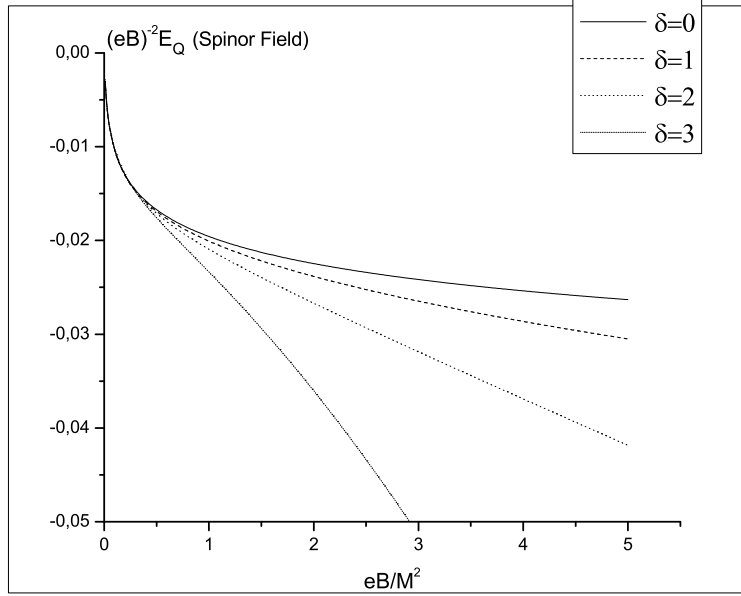


Figure 1: $(2^{[D/2]-2})^{-1}(eB)^{-2}E_Q$ versus eB/M^2 , for fixed ratio $M/m_f = 10^3$, and $\delta = 0, 1, 2, 3$, for the fermion field.

6 Numerical results for the quantum energy

In Fig. 1 we have plotted the $(2^{[D/2]-2})^{-1}(eB)^{-2}E_Q$ as a function of eB/M^2 , in the case of fermions. This figure confirms numerically the strong field asymptotic behavior for the quantum energy, for $\delta = 1, 2, 3$ (see Eq. (37)). In addition, for $eB \ll M^2$ the quantum energy is independent from the extra dimensions and it coincides with the four dimensional result, as is required. Thus the existence of the extra dimensions is not observable for weak magnetic fields $eB \ll M^2$. The corresponding case for the scalar fields is presented in Fig. 2, and as we see, it has the same features with that of the fermion fields (see also Eq. (49)).

In Fig. 3 we have plotted $(2^{[D/2]-2})^{-1}(eB)^{-2}E_Q$ as a function of eB/M^2 for the spinor field, in the case of $\delta = 0, 4, 5, 6$. From the figure we observe that the extra dimensions does not alter the four dimensional quantum energy in the limit $eB \ll M^2$, as is required. Note that for $eB \gg M^2$ the quantum energy for $\delta = 4, 5, 6$ is positive, contrary to the case of $\delta = 1, 2, 3$. The corresponding figure, in the case of the scalar fields, has the same features with those of Fig. 3.

Finally, in Fig. 4, we present a typical plot of the effective energy $E_{eff} = E_{class} + E_Q$ as a function of eB/M^2 , in the case of the fermion fields. The corresponding figure in the case of the scalar fields is not presented, as it exhibits exactly the same features with those of Fig. 4. Of course, Fig. 4 is not reliable for large values of the magnetic field, at which the perturbation theory breaks down (or $|E_Q| = E_{class}$). In addition, we remind the reader that the contributions of the possible higher dimensional operators has been completely ignored.

For the numerical computation we used the Eq. (36) for $\delta = 1, 2, 3$ and the Eq. (42) for

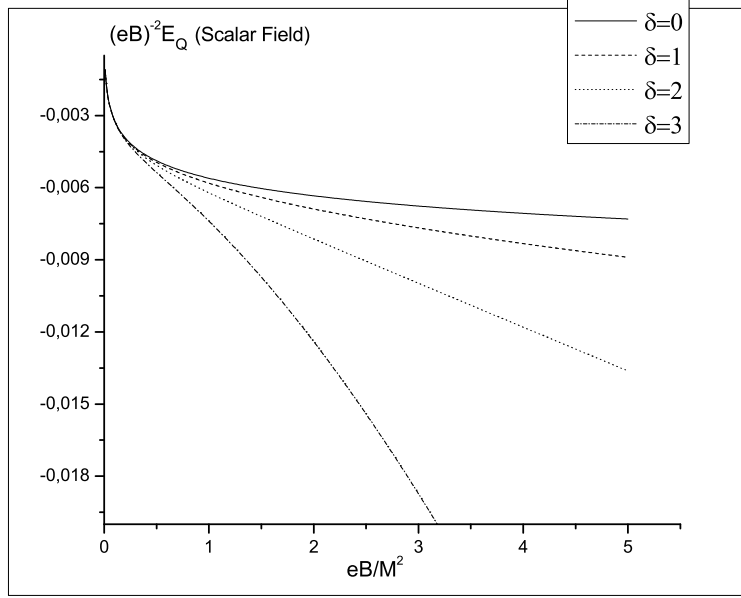


Figure 2: $(eB)^{-2}E_Q$ versus eB/M^2 , for fixed ratio $M/m_s = 10^3$, and $\delta = 0, 1, 2, 3$, for the scalar field.

$\delta = 4, 5, 6$.

We would like to estimate for which magnetic fields the quantum energy becomes comparable with the quantum energy. In Table 2 we determine the magnetic fields for which $|E_Q| = 0.1E_{class}$ and $|E_Q| = E_{class}$.

In the case of four dimensions, the strong field ($B \gg e/m_f^2 \sim 10^{14}G$ for $m_f = 0.5MeV$ and $e^2/4\pi = 1/137$) asymptotic behavior of the quantum energy is given by the formula $E_Q = -\frac{(eB)^2}{24\pi^2} \ln(\frac{eB}{m_f^2})$ (see Eq. (24)). The quantum energy is equal to the ten per cent of the classical energy (or $|E_Q| = 0.1E_{class}$) for magnetic fields of the order $B \sim 10^{70}G$. The value of the magnetic field for which the perturbation theory breaks down (or $|E_Q| = E_{class}$) is $B \sim 10^{574}G$. These values are entirely unrealistic. Of course, this is due to the small

$ E_Q =$	M	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$
$0.1E_{class}$	1 Tev	$B \sim 10^{70}G$	$B \sim 8.6 \cdot 10^{29}G$	$B \sim 9.5 \cdot 10^{27}G$	$B \sim 2.1 \cdot 10^{27}G$
E_{class}	1 Tev	$B \sim 10^{574}G$	$B \sim 8.6 \cdot 10^{31}G$	$B \sim 9.5 \cdot 10^{28}G$	$B \sim 9.5 \cdot 10^{27}G$
$0.1E_{class}$	20 Tev	$B \sim 10^{70}G$	$B \sim 3.5 \cdot 10^{32}G$	$B \sim 3.8 \cdot 10^{30}G$	$B \sim 8.5 \cdot 10^{29}G$
E_{class}	20 Tev	$B \sim 10^{574}G$	$B \sim 3.5 \cdot 10^{34}G$	$B \sim 3.8 \cdot 10^{31}G$	$B \sim 3.9 \cdot 10^{30}G$

Table 2: We present a comparison between the quantum and the classical energy, and in particular we determine the values of the magnetic fields for which $|E_Q| = 0.1E_{class}$ and $|E_Q| = E_{class}$ for $\delta = 0, 1, 2, 3$, $M = 1TeV, 20TeV$, $e^2/4\pi = 1/137$, and $m_f = 0.5MeV$. We have not included the cases of $\delta = 4, 5, 6$, as the value of w_R is not known.

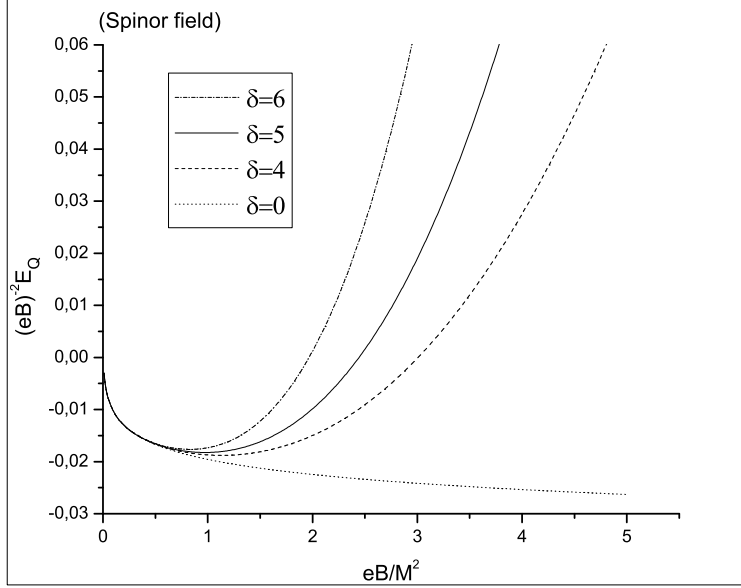


Figure 3: $(2^{[D/2]-2})^{-1}(eB)^{-2}E_Q$ versus eB/M^2 , for fixed ratio $M/m_f = 10^3$, $\delta = 0, 4, 5, 6$, for the fermion field.

coupling constant of QED, and the weak logarithmic growth of the quantum energy.

In the case of extra dimensions, and in particular for $\delta = 3$ (for $\delta = 1, 2$ see Table 2), we see that the quantum energy becomes equal to the ten per cent of the classical energy for $B \sim 10^{27}G$ ($M = 1TeV$), and the perturbation theory breaks down for $B \sim 10^{28}G$. These values are significantly smaller than the corresponding values of the four dimensional case, but they are still very large. For example the magnetic fields in neutron stars are of the order of $B \sim 10^{14}G$, and the magnetic fields which might have been created during the electroweak phase transition are of the order of $B \sim 10^{24}G$ [24], which are smaller than $B \sim 10^{27}G$ where the effects of the extra dimensions become significant. However, the authors of Ref. [25], have assumed the existence of magnetic fields $B \sim 10^{33}G$, in the early universe, in order to explain the present-day galactic magnetic fields. Thus, for magnetic fields of this order of magnitude the effects of the extra dimensions should be taken into account (see also the discussion in the next section).

7 Conclusions and Discussion

We considered a possible generalization of a U(1) gauge field theory with fermion fields (or scalar fields) in the case of δ extra compact dimensions, and we presented a computation for the fermion induced quantum energy, in the presence of a constant magnetic field which is directed towards the x_3 axis.

Moreover, we would like to note that realist magnetic fields can not cover an infinite volume. However, in the strong field limit the typical length scale $1/\sqrt{eB}$ of the magnetic

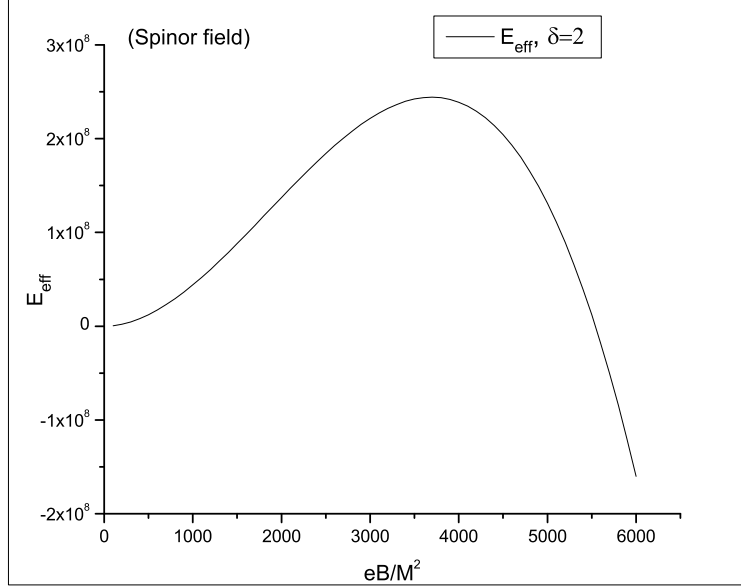


Figure 4: $E_{eff} = E_{class} + E_Q$ versus eB/M^2 , for fixed ratio $M/m_f = 210^6$, $\delta = 2$ and $e^2/4\pi = 1/137$ for the fermion field.

field is much smaller than the spatial size L , or $eB L^2 \gg 1$, and in this case the magnetic field behaves as if it was constant everywhere. Thus, we expect the formulas we derive to be valid, even if the magnetic field vanishes outside a large volume $V = L^3$.

The model we examined is nonrenormalizable, and it should be assumed as a low energy effective theory with a large physical cut-off Λ_{Ph} (Wilsonian effective field theory). A remarkable feature is that the one loop result for the quantum energy is independent of the cut-off Λ_{Ph} for small δ (or $\delta = 1, 2, 3$). However, for $\delta = 4, 5, 6$ the nonrenormalizable character of the model worsens and the scale Λ_{Ph} (or the proper time cut-off Λ) appears explicitly in the quantum energy. In this case we included a higher dimension operator of the form $w_D/2 \int d^D x (F_{\mu\nu} F^{\mu\nu})^2$ in the Lagrangian of the model. Then, the cut-off dependent part of Eq. (40) can be incorporated in this term according to the renormalization conditions of Eq. (41). Thus, in the case of $\delta = 4, 5, 6$, the effective energy contains a term of the form $\frac{w_R}{2} B^4$, with an additional free parameter w_R .

We would like to note that there are alternate ways to treat this difficulty. For example, we can ignore all the higher dimension operators and assume that the proper time cut-off Λ is not just a typical regularization parameter, but it is connected with the physical cut-off Λ_{Ph} via a linear relation (see Ref. [12]). Then, the result for the effective energy, for $\delta = 4, 5, 6$, will be a finite quantity which depends on the physical cut-off Λ_{Ph} (see Eq. (40)). However, in this paper, we adopt the philosophy which is presented in Ref. [20] and we incorporate the cut-off dependent term in a higher dimension operator, as we explain in the previous paragraph and in section 4.

We also studied the effect of the extra dimensions on the strong field ($eB \gg M^2$)

asymptotic behavior of the effective energy. We see that there is a critical value of the magnetic field $B_{cr} = M^2/e$ ($\sim 10^{26}G$ for $M=1$ Tev), above which the extra dimensions behave as if they were noncompact (see the figures in section 6), and as a consequence the quantum energy increases rapidly according to a power law behavior (see Eqs. (37),(43) and (45)). This behavior modifies the weak logarithmic growth of the quantum energy in the case of four dimensions (see Eq. (24)).

In this work we have not examined the interesting case of massless fermions. We preferred to assume that the fermions are massive in order to make contact with the Schwinger formula for four dimensions, which has been formulated only for massive fermions. However, our results for the strong field asymptotic behavior are valid even for massless fermions. In this case the mass of the fermions m_f is viewed as an infrared cut-off parameter. An interesting feature of the strong field asymptotic formulas, of Eq. (32), (37), (43) and (45), is that they are independent of the infrared cut-off m_f , for $eB \gg M^2 \gg m_f^2$.

It is interesting to note that in higher dimensional spaces constant magnetic fields are characterized by the electromagnetic tensor $F_{\mu\nu}$ where $\mu, \nu = 1, 2, \dots, D-1$, which has more than one independent components (see for example Ref. [16]). These more general magnetic fields are physical objects which are introduced by the higher dimensional theory. Even these fields have not been observed directly in low energy physics, they can be relevant physically indirectly. For example, in Ref. [26], the existence of a magnetic field in the intrinsic space can be used as a tool for supersymmetry breaking. In this paper, we have assumed the special case of a magnetic field which is directed toward the x_3 axis of the physical space. The computation of the quantum energy in the most general case of a higher dimensional magnetic field is beyond the scope of this paper and could be a topic of further investigation.

We emphasize again, that this extra-dimension $U(1)$ model with massive fermions (or with a massive scalar field), can not be viewed as an extra-dimension extension of QED. It is only a toy model for the study of the effects of the extra dimensions. Thus, the results of this paper are only suggestive and not realistic.

However, we can use the $U(1)$ model in order to extract results for a more realistic case. We will assume an extra-dimension version of the standard model, before the electroweak symmetry ($SU_L(2) \times U_Y(1)$) breaking. Then the $U(1)$ symmetry, of the $U(1)$ model of this work, corresponds to the $U_Y(1)$ of the standard model, and the magnetic field corresponds to a hypercharge $U_Y(1)$ magnetic field. Then the effective energy, per unit volume, for $\delta = 4, 5, 6$, in the case of $U_Y(1)$ reads:

$$E_{eff} = \frac{1}{2g_R^2}(g_RB)^2 + \frac{w_R}{2g_R^4}(g_RB)^4 + E_Q$$

where g_R is the renormalized hypercharge $U_Y(1)$ coupling, and w_R is a free parameter (see section 4). The contributions to quantum energy are come from the spinor and scalar fields of the extra-dimension version of the standard model, and not from the vector fields. If we take into account that the main features of the quantum energy are the same for spinor and scalar fields (see sections 4 and 5), we expect the quantum energy E_Q , for $g_RB \gg M^2$ and $\delta = 4, 5, 6$, to be positive and to behave as $E_{eff}^{asympt} \sim (eB)^2(eB/M)^{\delta/2}$ (for $\delta = 4$ see Eq. (45)). Thus, for an appropriate negative value of the free parameter w_R it is possible for the effective energy to exhibit a minimum for $g_RB \gg M^2$, which corresponds to a stable hypercharge $U_Y(1)$ magnetic field. Note, that in the above mechanism we have completely

ignore the effects of the other possible higher order operators, and the negative value of the free parameter w_R has been put by hand.

It is worth to note that there are several scenarios for the generation of primordial magnetic fields, see for example Ref. [27]. Especially in the case of string cosmology, sufficiently large seeds for generating the observed galactic magnetic fields can be obtained from the amplification of electromagnetic vacuum fluctuations, due to the inflationary dynamics of the dilaton, see for example Ref. [28]. On the other hand, in the mechanism we present above, we assume that primordial magnetic fields may be created as the minimum of the one loop effective energy. However the magnetic fields that obtain from this mechanism, if we take into account the evolution of the early universe and the conservation of the magnetic flux (for details see Ref. [29]), can not give the correct size of the seed fields which are responsible for the generation of the present days observed galactic magnetic fields.

Finally, we emphasize that a construction of a reliable scenario for the generation of primordial magnetic fields is beyond the scope of this paper. Our main motivation is to investigate a classical topic like the Heisenberg Euler lagrangian in the case of extra dimensions, and to note the rapid power growing of the quantum energy in this case.

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